AP Calculus BC
Solutions – FR Questions

1981 BC3

(a) $t=1 \Rightarrow \dfrac{t}{1+t}=\dfrac{1}{2}$

Then $S=\sum_{n=0}^{\infty}\dfrac{1}{2^n}=\dfrac{1}{1-\frac{1}{2}}=2$

(b) Since S is a geometric series, S converges if and only if $|r|=\left|\dfrac{t}{1+t}\right|<1$.

(i) We must have $|t|<|t+1|$. This means that the distance of t from 0 is less than the distance of t from –1. Therefore $t>-\dfrac{1}{2}$.

(c) $S(t)=\dfrac{1}{1-\frac{t}{1+t}}=1+t>10$ for $t>9$

1984 BC4

(a) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\frac{x^n n^n}{3^n n!}}\right|=\lim_{n\to\infty}\left|\frac{x}{3}\left(\frac{n+1}{n}\right)^n\right|=\left|\frac{x}{3}\cdot e\right|<1$$

Since the series converges for $|x|<\dfrac{3}{e}$, the radius of convergence is $\dfrac{3}{e}$.

(b) $$f(-1)=\sum_{n=1}^{\infty}\frac{(-1)^n n^n}{3^n n!}=-\frac{1}{3}+\frac{2}{9}-\frac{1}{6}+\ldots$$
$$\approx -\frac{5}{18}$$

(c) The series is alternating with the absolute value of the terms decreasing to 0. Therefore the error is less than the absolute value of the first omitted term. Hence

$$\left|f(-1)-\left(-\frac{5}{18}\right)\right|<\frac{4^4}{3^4 4!}=\frac{32}{243}$$

1986 BC5

(a) $f(x)=\sqrt{1+x}$ $\qquad f(0)=1$

$f'(x)=\frac{1}{2}(1+x)^{-1/2}$ $\qquad f'(0)=\frac{1}{2}$

$f''(x)=-\frac{1}{4}(1+x)^{-3/2}$ $\qquad f''(0)=-\frac{1}{4}$

$f'''(x)=\frac{3}{8}(1+x)^{-5/2}$ $\qquad f'''(0)=\frac{3}{8}$

$$T_f(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\dots.$$

(b) $T_g(x)=1+\frac{1}{2}x^3-\frac{1}{8}x^6+\frac{1}{16}x^9+\dots$

(c) Integrating the Taylor series in (b) gives

$T_h(x)=C+x+\frac{1}{8}x^4-\frac{1}{56}x^7+\dots$

$h(0)=4 \Rightarrow C=4$

$T_h(x)=4+x+\frac{1}{8}x^4-\frac{1}{56}x^7+\dots$

1997 BC2

(a) $f(4)=P(4)=7$

$\frac{f'''(4)}{3!}=-2,\ \ f'''(4)=-12$

(b) $P_3(x)=7-3(x-4)+5(x-4)^2-2(x-4)^3$

$P_3'(x)=-3+10(x-4)-6(x-4)^2$

$f'(4.3)\approx -3+10(0.3)-6(0.3)^2=-0.54$

(c) $P_4(g,x)=\int_4^x P_3(t)\,dt$

$=\int_4^x\left[7-3(t-4)+5(t-4)^2-2(t-4)^3\right]dt$

$=7(x-4)-\frac{3}{2}(x-4)^2+\frac{5}{3}(x-4)^3-\frac{1}{2}(x-4)^4$

(d) No. The information given provides values for $f(4), f'(4), f''(4), f'''(4)$ and $f^{(4)}(4)$ only.

2001 BC6

(a) $$\lim_{n\to\infty}\left|\frac{\dfrac{(n+2)x^{n+1}}{3^{n+2}}}{\dfrac{(n+1)x^n}{3^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)}{(n+1)}\frac{x}{3}\right| = \left|\frac{x}{3}\right| < 1$$

At $x=-3$, the series is $\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{3}$, which diverges.

At $x=3$, the series is $\sum_{n=0}^{\infty}\frac{n+1}{3}$, which diverges.

Therefore, the interval of convergence is $-3 < x < 3$.

(b) $$\lim_{x\to 0}\frac{f(x)-\dfrac{1}{3}}{x} = \lim_{x\to 0}\left(\frac{2}{3^2}+\frac{3}{3^3}x+\frac{4}{3^4}x^2+\cdots\right) = \frac{2}{9}$$

(c) $$\int_0^1 f(x)\,dx = \int_0^1\left(\frac{1}{3}+\frac{2}{3^2}x+\frac{3}{3^3}x^2+\cdots+\frac{n+1}{3^{n+1}}x^n+\cdots\right)dx$$
$$= \left(\frac{1}{3}x+\frac{1}{3^2}x^2+\frac{1}{3^3}x^3+\cdots+\frac{1}{3^{n+1}}x^{n+1}+\cdots\right)\Bigg|_{x=0}^{x=1}$$
$$= \frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n+1}}+\cdots$$

(d) The series representing $\int_0^1 f(x)\,dx$ is a geometric series.

Therefore, $\int_0^1 f(x)\,dx = \dfrac{\dfrac{1}{3}}{1-\dfrac{1}{3}} = \dfrac{1}{2}$.